BIRATIONAL GEOMETRY OF IRREGULAR VARIETIES IN ZERO AND POSITIVE CHARACTERISTIC

by

Yuan Wang

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STATEMENT OF DISSERTATION APPROVAL

The dissertation of **Yuan Wang**
has been approved by the following supervisory committee members:

Christopher Hacon,	Chair(s)	**1 Mar. 2017** Date Approved
Aaron Bertram,	Member	**1 Mar. 2017** Date Approved
Tommaso de Fernex,	Member	**1 Mar. 2017** Date Approved
Andrejs Treibergs,	Member	**1 Mar. 2017** Date Approved
Ilya Zharov,	Member	**1 Mar. 2017** Date Approved

by **Peter Trapa**, Chair/Dean of
the Department/College/School of **Mathematics**
and by **David B. Kieda**, Dean of The Graduate School.

ABSTRACT

In this dissertation, we mainly focus on constructing two results that characterize certain varieties by their birational invariants. In characteristic 0, we generalize a celebrated theorem of Kawamata by showing that for a projective log canonical pair (X, Δ), if the Kodaira dimension of $K_X + \Delta$ is 0 and the dimension of the Albanese variety $\mathrm{Alb}(X)$ of X is equal to the dimension of X, then X is birational to an abelian variety. In characteristic $p > 0$, we show a classification result for surfaces of general type beyond the Noether line. More precisely, suppose that S is a minimal projective surface in characteristic $p \geq 11$, $\chi(\mathcal{O}_S) = 1$ and $\dim(\mathrm{Alb}(S)) = 4$, and S lifts to the second Witt vectors. Then under mild assumption on the Albanese variety and the Albanese morphism of S, S is a product of two smooth curves of genus 2.

CONTENTS

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CHAPTER 1

INTRODUCTION

The problem of characterizing varieties by their birational invariants is classical and yet of high current interest. It traces back to the Italian school of algebraic geometry, where this problem was well studied for complex surfaces which are smooth complex varieties in dimension 2.

In the Italian school, a well-known result due to Enriques is that given a minimal complex projective surface S, if the Kodaira dimension of S is 0 and the irregularity of S is 2, then S is an abelian surface. Much later, researchers found that such precise characterizations can be established in all dimensions. A celebrated result due to Kawamata is that for a smooth complex projective variety X, if the Kodaira dimension $\kappa(X)$ is 0 and the irregularity of X is equal to the dimension of X, then X is birational to an abelian variety ([17, Theorem 1 and Corollary 2]).

In this dissertation, we first prove a theorem which generalizes Kawamata's result to log canonical pairs.

Theorem A (Corollary 4.3)**.** *Let X be a normal projective variety over the complex number field $\mathbb{C}$ and (X, Δ) an lc pair. Assume that $\kappa(K_X + \Delta) = 0$ and the dimension of the Albanese variety of X is equal to the dimension of X. Then X is birational to an abelian variety.*

Another direction to extend the results of the Italian school is to work in positive characteristic. Following the work of Enriques, a classification of surfaces in both zero and positive characteristic was established and is widely recognized as the Enriques-Kodaira classification of surfaces. In the Enriques-Kodaira classification, surfaces with Kodaira dimension -1,0 and 1 are well understood. A detailed classification of surfaces of general type, however, seems to be very difficult. On the other hand, many useful properties that hold in characteristic 0, including many vanishing theorems, the stronger version of

Bertini's theorem as well as generic smoothness for fibrations, fail in the world of positive characteristic even in dimension 2. Therefore, the geometry of surfaces of general type is especially interesting and complicated.

From the geographical perspective, surfaces of general type on the Noether line are well understood in both characteristic 0 and characteristic p and are widely recognized as Horikawa surfaces ([12], [13], [23], [25]). However, those beyond the Noether line behave rather mysteriously. In particular, in characteristic p, they do not necessarily satisfy the Bogolomov-Miyaoka-Yau inequality. Recently, Langer ([21]) showed that the Bogomolov-Miyaoka-Yau inequality does hold for surfaces of general type that lift to the second Witt vectors. In the context of this theorem, using various techniques in derived categories, we show the following explicit classification result for surfaces of general type in characteristic p beyond the Noether line. This is the second main result of this thesis.

Theorem B (=Theorem 5.2)**.** *Let X be a minimal projective surface of general type over an algebraically closed field k. Denote the Albanese morphism of X as $a : X \to A$. Assume that*

- $\mathrm{char}(k) \geq 11$.
- *X is of maximal Albanese dimension, lifts to $W_2(k)$, and its Picard variety has no supersingular factors.*
- $\chi(\mathcal{O}_X) = 1$ *and* $\dim(A) = 4$.
- *a is separable.*

Then $X = C_1 \times C_2$ where C_1 and C_2 are smooth curves and $g(C_1) = g(C_2) = 2$.

CHAPTER 2

PRELIMINARITES

Throughout the dissertation, we work over an algebraically closed field k. As we will consider the cases in both characteristic 0 and characteristic $p > 0$, we do not make any restriction on the characteristic of k in advance.

2.1 Derived categories

For any scheme X of dimension n, we denote by $D(X)$ the derived category of $\mathcal{O}_X$-modules and denote by $D_c(X)$ (resp. $D_{qc}(X)$) the full subcategory of $D(X)$ consisting of complexes whose cohomologies are coherent (resp. quasi-coherent). We also denote the dualizing complex by $\omega_X^{\cdot}$ and define the dualizing functor D_X by $D_X(F) = R\mathcal{H}om(F, \omega_X^{\cdot}[n])$, $\forall F \in D_{qc}(X)$. We will use projection formula and Grothendieck duality in the following forms.

Theorem 2.1 (Projection formula)**.** *Let $f : X \to Y$ be a morphism of quasi-compact separated schemes. Let $F \in D_{qc}(X)$ be a sheaf and $G \in D_{qc}(Y)$ be a locally free sheaf. Then there is an isomorphism*

$$Rf_*(F) \otimes_{\mathcal{O}_Y} G \xrightarrow{\cong} Rf_*(F \otimes_{\mathcal{O}_X} f^*G).$$

Theorem 2.2 (Grothendieck duality)**.** *Let $f : X \to Y$ be a proper morphism of quasi-projective varieties, then*

$$Rf_* D_X(F) = D_Y(Rf_*(F)),\ \forall F \in D_{qc}(X).$$

We will also need Grauert-Riemenschneider vanishing theorem which is known to hold for smooth surfaces in any characteristic. We generalize the original statement in the following way:

Theorem 2.3 (Grauert-Riemenschneider vanishing theorem)**.** *Let $f : X \to Y$ be a projective and generically finite morphism, Y normal and quasi-projective and X a smooth surface. Then*

$R^1 f_*(\omega_X \otimes P) = 0$ *for any* $P \in \mathrm{Pic}^0(X)$.

Proof. The original statement and proof can be found in [19] as Theorem 2.20.1 which says that $R^1 f_* \omega_X = 0$. The proof also works for $\omega_X \otimes P$. □

2.2 Abelian varieties, Albanese map and the Fourier-Mukai transform

Definition 2.4. Let A be an abelian variety. For a subvariety $X \subseteq A$, we say that X *generates* A if X is not contained in a translate of any proper abelian subvariety of A.

Theorem-Definition 2.5. *Let X be a variety over k. There exists an abelian variety $\mathfrak{Alb}(X)$ together with a rational map $\mathfrak{a}_X : X \dashrightarrow \mathfrak{Alb}(X)$ such that*

- *$\mathfrak{a}_X(X)$ generates $\mathfrak{Alb}(X)$, i.e. $\mathfrak{a}_X(X)$ is not contained in a translate of any proper abelian subvariety of $\mathfrak{Alb}(X)$.*
- *For every rational map $f : X \dashrightarrow A$ from X to an abelian variety A, there exists a homomorphism $g : \mathfrak{Alb}(X) \to A$ and a constant $a \in A$ such that $f = g \circ \mathfrak{a}_X + a$.*

We call $\mathfrak{a}_X$ the Albanese map *of X and $\mathfrak{Alb}(X)$ the* Albanese variety via the Albanese map (AVAMa) *of X.*

Theorem-Definition 2.6. *Let X be a normal projective variety over k. There exists an abelian variety* $\mathrm{Alb}(X)$ *together with a morphism $a_X : X \to \mathrm{Alb}(X)$ such that*

- *$a_X(X)$ generates $\mathrm{Alb}(X)$, i.e. $a_X(X)$ is not contained in a translate of any proper abelian subvariety of $\mathrm{Alb}(X)$.*
- *For every morphism $f : X \to A$ from X to an abelian variety A, there exists a homomorphism $h : \mathrm{Alb}(X) \to A$ and a constant $b \in A$ such that $f = h \circ a_X + b$.*

We call a_X the Albanese morphism *of X and $\mathrm{Alb}(X)$ the* Albanese variety via the Albanese morphism (AVAMo) *of X.*

Throughout the article, unless otherwise stated, for a normal projective variety X, we use $\mathfrak{a}_X : X \dashrightarrow \mathfrak{Alb}(X)$ ($a_X : X \to \mathrm{Alb}(X)$) to denote the Albanese map (Albanese moprhism) of X. We refer to [20, Chapter II, §3] and [7, Chapter 9] for more details about the Albanese map and the Albanese morphism, respectively. The Albanese map and the Albanese morphism agree in characteristic 0 for normal proper varieties with rational singularities (cf. [34, Proposition 2.3] or [18, Lemma 8.1]). But they do not agree in general, as is illustrated by the following:

Example 2.7. Let X be a projective cone over any curve C of genus ≥ 1. Since X is covered by rational curves passing through the vertex and the Albanese morphism contracts all the rational curves, we see that the Albanese morphism of X has to be a morphism from X to a point. On the other hand, let X' be the blow-up of X at the vertex, $\nu : X \dashrightarrow X'$ the natural birational map, $p : X' \to C$ the natural $\mathbb{P}^1$ fibration from X' to C and $j : C \hookrightarrow \mathrm{Jac(C)}$ the embedding from C to its Jacobian. Then the Albanese map of X is $j \circ p \circ \nu$, whereas the Albanese morphism (and also the Albanese map) of X' is $j \circ p$.

Example 2.7 is also an example of the fact that the AVAMa is a birational invariant, but the AVAMo is not. Still, by definition, for a normal projective variety X, there exists a surjective homomorphism α from $\mathfrak{Alb}(X)$ to $\mathrm{Alb}(X)$ such that $a_X = \alpha \circ \mathfrak{a}_X$.

For convenience, in characteristic 0, we say that *the Albanese map of X is an algebraic fiber space* if the Albanese morphism of any smooth model of X is an algebraic fiber space.

Definition 2.8. Let X be a projective variety and let $a : X \to A$ be the Albanese morphism of X. We say that X is *of maximal Albanese dimension (mAd)* if $\dim(X) = \dim(a(X))$.

Let A be an abelian variety. Let $\hat{A}$ be its dual abelian variety and $p_A : A \times \hat{A} \to A$ and $p_{\hat{A}} : A \times \hat{A} \to \hat{A}$ be the projection morphisms. Let $\mathcal{P}$ be the Poincaré line bundle on $A \times \hat{A}$. We define the Fourier-Mukai transform $R\hat{S} : D(A) \to D(\hat{A})$ and $RS : D(\hat{A}) \to D(A)$ with respect to the kernel $\mathcal{P}$ by

$$R\hat{S}(\cdot) = Rp_{\hat{A},*}(p_A^*(\cdot) \otimes \mathcal{P}), \quad RS(\cdot) = Rp_{A,*}(p_{\hat{A}}^*(\cdot) \otimes \mathcal{P}).$$

Next we recall some facts proven in [27].

Theorem 2.9. *[27, Theorem 2.2] The following isomorphisms of functors hold on $D_{qc}(A)$ and $D_{qc}(\hat{A})$:*

$$RS \circ R\hat{S} = (-1_A)^*[-g]$$

$$R\hat{S} \circ RS = (-1_{\hat{A}})^*[-g]$$

where $[-g]$ means shifting by g steps to the right and -1_A means the inverse morphism on A.

Lemma 2.10. *[27, (3.1)] For any $x \in A$ and $y \in \hat{A}$, the following isomorphisms hold on $D_{qc}(A)$ and $D_{qc}(B)$, respectively:*

$$RS \circ T_y^* \cong (\otimes P_{-y}) \circ RS$$

$$RS \circ (\otimes P_x) \cong T_x^* \circ RS,$$

where $P_x = \mathcal{P}|_{\{x\}\times\hat{A}}$, $P_y = \mathcal{P}|_{A\times\{y\}}$ and T_x and T_y are translations by x and y on A and $\hat{A}$, respectively.

Lemma 2.11. *[27, (3.4)] Let A and B be abelian varieties, $\varphi : A \to B$ an isogeny and $\hat{\varphi} : \hat{B} \to \hat{A}$ the dual isogeny of φ. Then the following isomorphisms hold on $D_{qc}(B)$ and $D_{qc}(A)$, respectively:*

$$\varphi^* \circ RS_B = RS_A \circ \hat{\varphi}_*$$

$$\varphi_* \circ RS_A = RS_B \circ \hat{\varphi}^*.$$

Proposition 2.12. *[27, Proposition 3.11 (1)] Let A be an abelian variety, L an ample line bundle on A and $\phi_L : A \to \hat{A}$ the isogeny induced by L. Then*

$$\phi_L^*(\widehat{L}) = \bigoplus^{h^0(A,L)} L^\vee.$$

Recall the following

Definition 2.13. For an abelian variety A and a line bundle L on A, we define

$$K(L) = \{x \in A | T_x^*(L) \cong L\}$$

where T_x is the translation morphism with respect to x. We say that L is *nondegenerate* if $K(L)$ is finite; otherwise, we say that L is *degenerate*.

Theorem-Definition 2.14. *For any nondegenerate line bundle L on A by the vanishing theorem in [28, Section 16], there exists a unique $i \in \mathbb{Z}$, $0 \le i \le \dim(A)$ such that $H^i(X, L) \neq 0$ and we denote this i as $i(L)$.*

Proposition 2.15. *[26, Proposition (9.18)] $i(L) = 0$ for any ample line bundle L on A.*

Definition 2.16. Let A be an abelian variety. For a subvariety $X \subseteq A$, we say that X *generates* A if X is not contained in any proper abelian subvariety of A.

The next proposition is very useful in Chapter 5.

Proposition 2.17. *If L is a line bundle on an abelian variety A, then there is a unique integer $i \in \mathbb{Z}$, $0 \leq i \leq \dim(A)$ such that $R\hat{S}(L) = R^i\hat{S}(L)[-i]$ is a sheaf and its support is an abelian subvariety of $\hat{A}$. If L is nondegenerate, then this integer is equal to the integer $i(L)$ in Theorem-Definition 2.14.*

Proof. If L is nondegenerate, then by [28, p.145 Theorem], $i(L)$ can be computed as the number of positive roots of $P(n) = \chi(M^n \otimes L)$ for arbitrary ample line bundles M. Since for any P_y we have $\chi(M^n \otimes L) = \chi(M^n \otimes L \otimes P_y)$, we know that $i(L \otimes P_y)$ is independent of P_y. By cohomology and base change,

$$\begin{aligned} R^i\hat{S}(L) \otimes k(y) =& R^i p_{2,*}(p_1^*L \otimes \mathcal{P}) \otimes k(y) \cong H^i(A \times \{y\}, (p_1^*L \otimes \mathcal{P})|_{A\times\{y\}}) \\ =& H^i(A, L \otimes P_y). \end{aligned}$$

So $R^i\hat{S}(L) \neq 0$ if and only if $i = i(L)$. Therefore, we have

$$R\hat{S}(L) = R^{i(L)}\hat{S}(L)[-i(L)]$$

and it is supported on $\hat{A}$.

If L is degenerate, let $K(L)^0$ be the connected component of $K(L)$ containing the origin. Let $Z = (K(L)^0)_{\mathrm{red}}$ which by [26, Proposition 5.31] is an abelian subvariety of A. By Poincaré's complete reducibility theorem (cf. [28, p.160 Theorem]), there is an isogeny $\varphi : Y \times Z \to A$ where Y is an abelian subvariety of A. By the proof of [26, Proposition 9.27] there is a nondegenerate line bundle L_Z on Z such that $\varphi^*L = p_Z^*L_Z \otimes P_x$ for some $x \in \hat{Y} \times \hat{Z}$. By Lemma 2.10, Lemma 2.11 and [14, Exercise 5.13],

$$\begin{aligned} \hat{\varphi}_* R\hat{S}_A(L) =& R\hat{S}_{Y\times Z}(\varphi^*L) = R\hat{S}_{Y\times Z}(p_Z^*L_Z \otimes P_x) = T_x^* R\hat{S}_{Y\times Z}(p_Z^*L_Z) \\ =& T_x^* R\hat{S}_{Y\times Z}(L_Z \boxtimes \mathcal{O}_Y) = T_x^*(R\hat{S}_Z(L_Z) \boxtimes R\hat{S}_Y(\mathcal{O}_Y)) \\ =& T_x^*(R^{i(L_Z)}\hat{S}_Z(L_Z)[-i(L_Z)] \boxtimes R^{\dim(Y)}\hat{S}_Y(\mathcal{O}_Y)[-\dim(Y)]). \end{aligned}$$

By the fact that $\hat{\varphi}$ is finite, we get

$$R\hat{S}_A(L) = R^{i(L_Z)+\dim(Y)}\hat{S}_A(L)[-(i(L_Z) + \dim(Y))].$$

Observe that

$$0 \leq \dim(Y) \leq i(L_Z) + \dim(Y) \leq \dim(Z) + \dim(Y) \leq \dim(A).$$

Moreover, since $R^{i(L_Z)}\hat{S}_Z(L_Z)$ is supported on $\hat{Z}$ and $R^{\dim(Y)}\hat{S}_Y(\mathcal{O}_Y)$ is supported on the origin $e_{\hat{Y}}$ on $\hat{Y}$, we know that $R\hat{S}_A(L)$ is supported on $\hat{\varphi}^{-1} \circ T_x(\hat{Z} \times e_{\hat{Y}})$ which is an abelian subvariety in $\hat{A}$. □

Definition 2.18. For a nondegenerate line bundle L on an abelian variety A, we have seen that the integer constructed in Proposition 2.17 is compatible with the $i(L)$ in Theorem-Definition 2.14. So we define this integer to be the *index* of L and still use $i(L)$ to denote it. Moreover, we denote $R^{i(L)}\hat{S}(L)$, viewed as a sheaf on $\hat{A}$, by $\widehat{L}$.

Remark 2.19. Note that if L is an ample line bundle, then $\widehat{L}$ is locally free (see [30, Example 2.2] and [27, right before Corollary 2.4]).

Remark 2.20. In the situation of Definition 2.18, by [27, (3.8)], we have

$$\widehat{L}^\vee = (-1_A)^*\widehat{L^\vee}. \tag{2.1}$$

2.3 Lifting properties of algebraic varieties

For an algebraically closed field k of characteristic $p > 0$, let $W_2(k)$ be the ring of the second Witt vectors of k (for details see [6, Example 8.8]). Let X be a scheme over k and denote $S = \mathrm{Spec}(k)$ and $\widetilde{S} = \mathrm{Spec}(W_2(k))$. A *lifting* of X to $\widetilde{S}$ is a scheme $\widetilde{X}$, flat over $\tilde{S}$, such that $X = \widetilde{X} \times_{\widetilde{S}} S$. We say that X *lifts to* $W_2(k)$ or X *is liftable to* $W_2(k)$ if X has a lifting to $W_2(k)$.

We define X' by $X' = X \times_S S$ via the Frobenius morphism $F : S \to S$ and denote by F' the morphism $X \to X'$ naturally defined by $X \to S$ and the Frobenius morphism $F : X \to X$.

Theorem 2.21. *[4, Théorème 2.1] With the notation above, if X lifts to $W_2(k)$, then the following isomorphism holds in $D(X')$:*

$$\varphi : \bigoplus_{i<p} \Omega^i_{X'/S} \xrightarrow{\cong} \tau_{<p} F'_* \Omega^{\cdot}_{X/S}.$$

Moreover, the i-th cohomology of φ is the Cartier isomorphism for $i < p$.

2.4 Preliminaries on surfaces

Definition 2.22. Let X be a surface. We define the *geometric genus* $p_g(X)$ of X to be $h^0(X, \omega_X)$, the *irregularity* $q(X)$ of X to be $h^1(X, \mathcal{O}_X)$ and the *Euler characteristic* of X to be $\chi(\mathcal{O}_X)$. It is easy to see that for a surface X, we have $\chi(\mathcal{O}_X) = \chi(\omega_X)$.

Definition 2.23. For a variety X, we define the *Kodaira dimension* $\kappa(X)$ of X to be $\max_{m\geq 1}\{\dim(\phi_m(X))\}$. Here $\phi_m(X) : X \dashrightarrow \mathbb{P}H^0(X, \omega_X^m)$ is the rational map induced by $|mK_X|$. We say that $\kappa(X) = \infty$ if $|mK_X| = \emptyset$ for all $m \geq 1$.

Definition 2.24. An *irrational pencil of genus g* on a surface X is a fibration $p : X \to B$ where B is a smooth curve of genus $g \geq 1$.

We have the Bogomolov-Miyaoka-Yau inequality in positive characteristic established by Langer as follows.

Proposition 2.25. *[21, Theroem 13] Let X be a minimal minimal surface of general type. If* $\mathrm{char}(k) \geq 3$ *and X is liftable to $W_2(k)$, then*

$$K_X^2 \leq 9\chi(\mathcal{O}_X).$$

The following proposition is well known, but since its proof is very short, we would still like to include it.

Proposition 2.26. *If S is an elliptic surface over k, then S is not of general type.*

Proof. Let $f : S \to B$ be an elliptic fibration of S and E a general fiber. If S is of general type, by the adjunction formula,

$$K_E = (K_S + E)|_E = K_S|_E$$

is big. However, since E is an elliptic curve, we know $K_E = \mathcal{O}_E$ is not big. Contradiction. □

2.5 Singularities in birational geometry

We present some basic definitions for the singularities in birational geometry, which can be found in [9]. Let X be a normal variety and Δ is an effective $\mathbb{Q}$-divisor on X. We say that (X, Δ) is a *pair* if $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

A *log resolution* of (X, Δ) is a proper birational morphism $f : Y \to X$ such that Y is smooth and $f_*^{-1}\Delta + \mathrm{Ex}(f)$ is simple normal crossing. We can write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where Γ and E are effective $\mathbb{Q}$-divisors with no common components. For any divisor F on Y, we define the *discrepancy* of F as $a(F; X, \Delta) := \mathrm{mult}_F(E - \Gamma)$. Note that the discrepancy is independent of the log resolution. We say that (X, Δ) is *log canonical (lc)* if $a(F; X, \Delta) \geq -1$.

2.6 Some facts about torsion-free sheaves

Lemma 2.27. *Let Z be an irreducible variety and suppose that L is a line bundle and $\mathcal{F}$ is a torsion-free coherent sheaf on Z. If $\varphi : L \to \mathcal{F}$ is a nonzero morphism, then φ is injective.*

Proof. We know that $\mathrm{Supp}(\mathrm{im}(\varphi))$ is a closed subset. On the other hand, $\mathcal{F}$ is torsion-free so it cannot contain a torsion sheaf, so $\mathrm{Supp}(\mathrm{im}(\varphi)) = Z$. Next we restrict the morphism to an affine open set $U = \mathrm{Spec}(A)$ and suppose $L|_U = \tilde{M} = (\tilde{Ae})$ and $\mathcal{F}|_U = \tilde{N}$ where M and N are A-modules and we denote by $\phi : M \to N$ the induced homomorphism. Suppose $\varphi(e) = n \in N$, then by the above argument, $n \neq 0$. If there exists a nonzero a in A such that $\varphi(ae) = a \cdot n = 0$, then it contradicts the fact that N is torsion-free. □

Lemma 2.28. *Let X and Y be projective varieties and $f : X \to Y$ a finite morphism. If $\mathcal{F}$ is a torsion-free sheaf on Y, then $f^*\mathcal{F}$ is also torsion-free.*

Proof. We may assume that X and Y are affine and f is induced by a ring homomorphism $\psi : R \to S$ where $X = \mathrm{Spec}(S)$, $Y = \mathrm{Spec}(R)$.

Now the claim becomes that if M is a torsion-free R-module, then $M \otimes_R S$ is a torsion-free S-module. Since f is finite, we can assume that S is generated by $\{s_i\}$ as an R-module. If the claim is not true, then there exists $0 \neq \sum_i m_i \otimes s_i \in M \otimes_R S$ and $0 \neq s' \in S$ such that

$$s'(\sum_i m_i \otimes s_i) = \sum_i m_i \otimes s_i s' = 0.$$

By [5, Lemma 6.4], there exist $m'_j \in M$ and $a_{ij} \in R$ such that

$$\sum_j a_{ij} m'_j = m_i, \forall i$$

and

$$\sum_i a_{ij} s_i s' = 0, \forall j$$

Since $s' \neq 0$ and S is integral, by assumption, it follows that $\sum_i a_{ij} s_i = 0$. By [5, Lemma 6.4], again we get $\sum_i m_i \otimes s_i = 0$ which is a contradiction. □

CHAPTER 3

GENERIC VANISHING

In this chapter, we will review some known results on generic vanishing in characteristic 0, and construct a generic vanishing for surfaces that lift to the second Witt vectors in characteristic $p > 0$.

3.1 Characteristic 0

Definition 3.1. Let A be an abelian variety of dimension n and $\mathcal{F}$ a coherent sheaf on A. We define $V^i(\mathcal{F})$, the *cohomology support loci*, as

$$V^i(\mathcal{F}) = \{P \in \mathrm{Pic}^0(X) | h^i(A, \mathcal{F} \otimes P) > 0\}.$$

$\mathcal{F}$ is said to *satisfy Generic Vanishing with index $-k$*, or to be *GV_{-k}*, if $\mathrm{codim}_{\mathrm{Pic}^0(X)} V^i(\mathcal{F}) \geq i - k$ for all $i \geq 0$.

The following generic vanishing theorem can be deduced from [8, Theorem 1.2] and [31, Theorem A].

Theorem 3.2. *With the notation as in 3.1, the following are equivalent:*
(1) $\mathcal{F}$ is GV_0.
(2) $H^i(A, \mathcal{F} \otimes \widehat{L}^{\vee}) = 0$ for any $i > 0$ and any sufficiently ample line bundle L on X.
(3) $R\hat{S}(D_A(\mathcal{F})) = R^0\hat{S}(D_A(\mathcal{F}))$.

Remark 3.3. In this paper, we use the notion of "sufficiently ample" line bundle on a variety X to mean, given any ample line bundle L, a power $L^{\otimes m}$ with $m \gg 0$.

Theorem 3.4. *[33, Variant 5.5] Let $f : X \to A$ be a morphism from a normal projective variety to an abelian variety. If (X, Δ) is a lc pair and $k > 0$ is any integer such that $k(K_X + \Delta)$ is Cartier, then $f_*\mathcal{O}_X(k(K_X + \Delta))$ is a GV-sheaf.*

The following fact due to Pink and Roessler (of [32]) will also play an important role.

Proposition 3.5. *Let X be a projective variety of dimension n over k and assume that the Picard variety of X has no supersingular factors. Let*

$$S^{i,j}_m(X) = \{P \in \mathrm{Pic}^0(X) | h^{i,j}(X,P) \geq m\}$$

for any $i, j, m \geq 0$. Then $S^{i,j}_m$ is completely linear, i.e. its irreducible components are translates of abelian subvarieties by torsion elements in $\mathrm{Pic}^0(X)$. In particular, $V^1(\omega_X) = S^{1,n}_1(X)$ is completely linear.

Proof. By [32, Proposition 4.1 and Proposition 3.1]. □

3.2 Characteristic $p > 0$

It is known that generic vanishing fails in general in positive characteristic (cf. [10]). In this section, we construct the following generic vanishing theorem in dimension 2.

Theorem 3.6. *Let X be a smooth projective surface over an algebraically closed field k of positive characteristic, A an abelian variety and $a : X \to A$ a generically finite morphism. If X lifts to $W_2(k)$, then $H^i(X, \Omega^j_X \otimes P \otimes a^*\widehat{L}^\vee) = 0$ for any $i + j \geq 3$, $P \in \mathrm{Pic}^0(X)$ and ample line bundle L on $\hat{A}$. In particular, for any $k > 0$ and $P \in \mathrm{Pic}^0(X)$, $H^k(A, a_*(\omega_X \otimes P) \otimes Q) = 0$ for general $Q \in \mathrm{Pic}^0(A)$.*

Remark 3.7. In particular, in Theorem 3.6, if we let $P = \mathcal{O}_X$, we get $H^k(A, a_*\omega_X \otimes Q) = 0$ for $k \geq 1$ and general $Q \in \mathrm{Pic}^0(A)$. By Theorem 3.2, this is equivalent to $H^k(X, \omega_X \otimes a^*Q) = 0$, in particular, $V^1(\omega_X) \neq \mathrm{Pic}^0(X)$. Then by applying the semicontinuity theorem (cf. [11, Ch. III Theorem 12.8]) with $f : X \times \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$ and $\mathscr{F} = K_{X\times \mathrm{Pic}^0(X)} \otimes \mathcal{P}$, where $\mathcal{P}$ is the Poincaré line bundle on $X \times \mathrm{Pic}^0(X)$, we get that $V^1(\omega_X)$ is a proper closed subset of $\mathrm{Pic}^0(X)$. This means that $\mathrm{codim}_{\mathrm{Pic}^0(X)} V^1(\omega_X) \geq 1$. Moreover, by Serre duality, $H^2(X, \omega_X \otimes P) = H^0(X, P^\vee)^\vee$ and it is nonzero iff $P = \mathcal{O}_X$, so we also have $\mathrm{codim}_{\mathrm{Pic}^0(X)} V^2(\omega_X) \geq 2$. Therefore, we see that Theorem 3.6 actually implies that ω_X is GV_0.

The proof of Theorem 3.6 is in the spirit of the article of Deligne and Illusie [4]. Let $\hat{A}$ be the dual abelian variety of A, L be an ample line bundle on $\hat{A}$ and $\phi_L : \hat{A} \to A$ be the isogeny induced by L. We use F to denote the k-linear Frobenius morphism. To prove the theorem, we need the following lemmas.

Lemma 3.8. *Suppose $\mathcal{F}$ is a coherent sheaf on A. Then there exists an e_0 such that for any $e \geq e_0$, $H^i(A, \mathcal{F} \otimes F^{e,*}(\widehat{L}^\vee) \otimes Q) = 0$ for any $i > 0$ and $Q \in \mathrm{Pic}^0(A)$.*

Proof. By Fujita's vanishing theorem (cf. [22, Theorem 1.4.35]) and Proposition 2.12, there exists an e_0 such that for any $e \geq e_0$,

$$\begin{aligned} 0 &= H^i(\hat{A}, \phi_L^*(\mathcal{F}) \otimes (\bigoplus^{h^0(\hat{A},L)} L^{p^e}) \otimes Q) \\ &= H^i(\hat{A}, \phi_L^*(\mathcal{F}) \otimes F^{e,*}\phi_L^*(\widehat{L}^\vee) \otimes Q) \\ &= H^i(\hat{A}, \phi_L^*(\mathcal{F}) \otimes \phi_L^* F^{e,*}(\widehat{L}^\vee) \otimes Q) \\ &= H^i(\hat{A}, \phi_L^*(\mathcal{F} \otimes F^{e,*}(\widehat{L}^\vee)) \otimes Q). \end{aligned}$$

for any $i > 0$ and $Q \in \mathrm{Pic}^0(\hat{A})$. Then by cohomology and base change and Lemma 2.11, we know that

$$RS(\phi_L^*(\mathcal{F} \otimes F^{e,*}(\widehat{L}^\vee))) = \widehat{\phi_L}_* R\hat{S}(\mathcal{F} \otimes F^{e,*}(\widehat{L}^\vee)) \tag{3.1}$$

is a sheaf in degree 0. Moreover, since $\widehat{\phi_L}$ is finite, then $R\hat{S}(\mathcal{F} \otimes F^{e,*}(\widehat{L}^\vee))$ is also a sheaf in degree 0. By cohomology and base change, this implies that $H^i(A, \mathcal{F} \otimes F^{e,*}(\widehat{L}^\vee) \otimes Q) = 0$ for any $i > 0$ and $Q \in \mathrm{Pic}^0(A)$. □

Lemma 3.9. *Let P be a line bundle in $\mathrm{Pic}^0(X)$. We denote the support of $\phi_L^*(a_*(\Omega_X^1 \otimes P))$ by Z. Then $\phi_L^*(a_*(\Omega_X^1 \otimes P))$ is a torsion-free sheaf on Z.*

Proof. $\Omega_X^1 \otimes P$ is torsion-free by definition, and $a_*(\Omega_X^1 \otimes P)$ is torsion-free on $a(X)$ because any push-forward of a torsion-free sheaf is torsion-free on the image. Then $\phi_L^*(a_*(\Omega_X^1 \otimes P))$ is torsion-free on Z by Lemma 2.28 and finiteness of ϕ_L. □

Lemma 3.10. *There exists e_0 such that for any $e \geq e_0$, $i > 0$ and $P \in \mathrm{Pic}^0(X)$, $H^i(X, \omega_X \otimes P \otimes F^{e,*}a^*\widehat{L}^\vee) = 0$ and $H^2(X, \Omega_X^1 \otimes P \otimes F^{e,*}a^*\widehat{L}^\vee) = 0$.*

Proof. First we prove that $H^i(X, \omega_X \otimes P \otimes F^{e,*}a^*\widehat{L}^\vee) = 0$. By Theorem 2.3, we have $R^i a_*(\omega_X \otimes P) = 0$ for any $i > 0$. Hence, we have

$$H^i(X, \omega_X \otimes P \otimes F^{e,*}a^*\widehat{L}^\vee) = H^i(X, \omega_X \otimes P \otimes a^*F^{e,*}\widehat{L}^\vee) = H^i(A, a_*(\omega_X \otimes P) \otimes F^{e,*}\widehat{L}^\vee)$$

where the first equality is by the commutativity of a and F and the second equality is by the Projection Formula and degeneration of a Leray spectral sequence (cf. [11, Exercise

III.8.1]). The claim then follows from Lemma 3.8 after we replace $\mathcal{F}$ and Q by $a_*(\omega_X \otimes P)$ and $\mathcal{O}_A$, respectively.

Next we will prove that $H^2(X, \Omega^1_X \otimes P \otimes F^{e,*}a^*\widehat{L}^\vee) = 0$ for $e \gg 0$ which by Serre duality is equivalent to $H^0(X, \Omega^1_X \otimes P^\vee \otimes F^{e,*}a^*\widehat{L}) = 0$. We first prove that there exists an e_0 such that for any $e \geq e_0$ and any $Q \in \mathrm{Pic}^0(\hat{A})$, $H^0(\hat{A}, \phi_L^*(a_*(\Omega^1_X \otimes P^\vee)) \otimes (L^{-p^e}) \otimes Q) = 0$. This is because every nonzero element in $H^0(\hat{A}, \phi_L^*(a_*(\Omega^1_X \otimes P^\vee)) \otimes (L^{-p^e}) \otimes Q)$ corresponds to a nonzero morphism $L^{p^e} \otimes Q^\vee \to \phi_L^*(a_*(\Omega^1_X \otimes P^\vee))$. We claim that this morphism is injective after we restrict it to an irreducible component Z_0 of Z. Indeed, $L^{p^e} \otimes Q^\vee|_{Z_0}$ is a line bundle which is torsion-free of rank 1 and $\phi_L^*(a_*(\Omega^1_X \otimes P^\vee))|_{Z_0}$ is torsion-free by Lemma 3.9, so the injectivity follows from Lemma 2.27. Since $\dim(Z_0) = 2$, after taking $H^0(Z_0, \cdot)$ on both sides of the morphism $L^{p^e} \otimes Q^\vee|_{Z_0} \to \phi_L^*(a_*(\Omega^1_X \otimes P^\vee))|_{Z_0}$, we see that for any $Q \in \mathrm{Pic}^0(X)$, $h^0(Z_0, L^{p^e} \otimes Q^\vee|_{Z_0})$ goes to infinity as $e \gg 0$ while $h^0(Z_0, \phi_L^*(a_*(\Omega^1_X \otimes P^\vee)|_{Z_0}))$ is constant and the induced homomorphism is still injective. This is a contradiction which completes the proof that $H^0(\hat{A}, \phi_L^*(a_*(\Omega^1_X \otimes P^\vee)) \otimes ((-L)^{p^e}) \otimes Q) = 0$.

Now by cohomology and base change, $H^0(\hat{A}, \phi_L^*(a_*(\Omega^1_X \otimes P^\vee)) \otimes ((-L)^{p^e}) \otimes Q) = 0$ implies that $R^0S(\phi_L^*(a_*(\Omega^1_X \otimes P^\vee) \otimes F^{e,*}(\widehat{L}))) = 0$. By Lemma 2.11, we see that

$$RS(\phi_L^*(a_*(\Omega^1_X \otimes P^\vee) \otimes F^{e,*}(\widehat{L}))) = (R\widehat{\phi_L}_* \circ R\hat{S})(a_*(\Omega^1_X \otimes P^\vee) \otimes F^{e,*}(\widehat{L})),$$

and after taking cohomology in degree 0, we have

$$0 = R^0(\widehat{\phi_L}_* \circ \hat{S})(a_*(\Omega^1_X \otimes P^\vee) \otimes F^{e,*}(\widehat{L})).$$

Since $\widehat{\phi_L}$ is finite, the following Grothendieck spectral sequence

$$\begin{aligned} E_2^{ij} &= R^i\widehat{\phi_L}_* R^j\hat{S}(a_*(\Omega^1_X \otimes P^\vee) \otimes F^{e,*}(\widehat{L})) \\ &\Rightarrow R^{i+j}(\widehat{\phi_L}_* \circ \hat{S})(a_*(\Omega^1_X \otimes P^\vee) \otimes F^{e,*}(\widehat{L})) \end{aligned}$$

degenerates at E_2, in particular

$$\begin{aligned} 0 =& R^0(\widehat{\phi_L}_* \circ \hat{S})(a_*(\Omega^1_X \otimes P^\vee) \otimes F^{e,*}(\widehat{L})) \\ =& R^0\widehat{\phi_L}_* R^0\hat{S}(a_*(\Omega^1_X \otimes P^\vee) \otimes F^{e,*}(\widehat{L})) \\ =& \widehat{\phi_L}_* R^0\hat{S}(a_*(\Omega^1_X \otimes P^\vee) \otimes F^{e,*}(\widehat{L})). \end{aligned}$$

Therefore, $R^0\hat{S}(a_*(\Omega^1_X \otimes P^\vee) \otimes F^{e,*}(\widehat{L})) = 0$, and then by cohomology and base change, $H^0(A, a_*(\Omega^1_X \otimes P^\vee) \otimes F^{e,*}\widehat{L} \otimes Q) = 0$ for all $Q \in U$ where U is an open set of $\mathrm{Pic}^0(A)$.

Now if we can show that $h^0(X, \Omega_X^1 \otimes P^\vee \otimes a^*(F^{e,*}\widehat{L} \otimes P))$ is constant with respect to P for $e \gg 0$, then we have

$$\begin{aligned} 0 &= H^0(A, a_*(\Omega_X^1 \otimes P^\vee) \otimes F^{e,*}\widehat{L} \otimes Q_0) \\ &= H^0(X, \Omega_X^1 \otimes P^\vee \otimes a^*(F^{e,*}\widehat{L} \otimes Q_0)) \\ &= H^0(X, \Omega_X^1 \otimes P^\vee \otimes a^*(F^{e,*}\widehat{L} \otimes Q)), \end{aligned}$$

for any $Q_0 \in U$ and any $Q \in \mathrm{Pic}^0(A)$, and after taking $P = \mathcal{O}_A$, we are done.

To prove that $h^0(X, \Omega_X^1 \otimes P^\vee \otimes a^*(F^{e,*}\widehat{L} \otimes Q))$ is constant with respect to Q, it suffices to prove that $h^1(X, \Omega_X^1 \otimes P^\vee \otimes a^*(F^{e,*}\widehat{L} \otimes Q))$ and $h^2(X, \Omega_X^1 \otimes P^\vee \otimes a^*(F^{e,*}\widehat{L} \otimes Q))$ are both constant. By Lemma 3.8, we know $H^i(A, a_*(\Omega_X^1 \otimes P^\vee) \otimes F^{e,*}\widehat{L}^\vee \otimes Q^\vee) = 0$ for any $i > 0, e \gg 0$ which implies that

$$\begin{aligned} &h^0(A, a_*(\Omega_X^1 \otimes P) \otimes F^{e,*}\widehat{L}^\vee \otimes Q^\vee) \\ =&h^0(X, \Omega_X^1 \otimes P \otimes a^*(F^{e,*}\widehat{L}^\vee \otimes Q^\vee)) \\ =&h^2(X, \Omega_X^1 \otimes P^\vee \otimes a^*(F^{e,*}\widehat{L} \otimes Q)) \end{aligned}$$

is constant. To prove that

$$\begin{aligned} &h^1(X, \Omega_X^1 \otimes P^\vee \otimes a^*(F^{e,*}\widehat{L} \otimes Q)) \\ =&h^1(X, \Omega_X^1 \otimes P \otimes a^*(F^{e,*}(\widehat{L}^\vee) \otimes Q^\vee)) \end{aligned}$$

is constant, we consider the Leray spectral sequence

$$\begin{aligned} E_2^{ij} &= H^i(A, R^j a_*((\Omega_X^1 \otimes P) \otimes a^*(F^{e,*}(\widehat{L}^\vee) \otimes Q^\vee))) \\ &= H^i(A, R^j a_*(\Omega_X^1 \otimes P) \otimes F^{e,*}(\widehat{L}^\vee) \otimes Q^\vee) \\ &\Rightarrow H^{i+j}(X, \Omega_X^1 \otimes P \otimes a^*(F^{e,*}(\widehat{L}^\vee) \otimes Q^\vee)) \end{aligned}$$

By Lemma 3.8, we know $E_2^{ij} = 0$ for $i > 0$, so the spectral sequence degenerates at E_2^{ij} and in particular

$$H^1(X, \Omega_X^1 \otimes P \otimes a^*(F^{e,*}(\widehat{L}^\vee) \otimes Q^\vee)) \cong H^0(A, R^1 a_*(\Omega_X^1 \otimes P) \otimes F^{e,*}(\widehat{L}^\vee) \otimes Q^\vee).$$

Since a is generically finite and X is a surface, $R^1 a_*(\Omega_X^1 \otimes P)$ is supported on the locus whose preimage with respect to a is 1-dimensional, that is, a finite number of points. So we know $H^0(X, R^1 a_*(\Omega_X^1 \otimes P) \otimes F^{e,*}(\widehat{L}^\vee) \otimes N)$ is constant for any line bundle N, in particular for any $Q \in \mathrm{Pic}^0(X)$. $\square$

Proof of Theorem 3.6. The rest of the proof follows [4, Lemme 2.9]. We fix an e_0 which satisfies the condition in Lemma 3.10, so after replacing P by $F^{e,*}P$, by projection formula and for dimensional reasons, we know that $H^j(X, F_{e,*}\Omega^i_X \otimes P \otimes a^*\widehat{L}^\vee) = 0$ for any $i + j \geq 3$, $e \geq e_0$ and $P \in \mathrm{Pic}^0(X)$. By Serre duality, this is equivalent to $H^j(X, F_{e,*}\Omega^i_X \otimes P \otimes a^*\widehat{L}) = 0$ for any $i + j \leq 1$ and $P \in \mathrm{Pic}^0(X)$. We consider the spectral sequence

$$E_1^{ij} = H^j(X, F_{e,*}\Omega^i_X \otimes P \otimes a^*\widehat{L}) \Rightarrow H^{i+j}(X, F_{e,*}\Omega^\cdot_X \otimes P \otimes a^*\widehat{L}).$$

This implies that

$$0 = H^i(X, F_{e,*}\Omega^\cdot_X \otimes P \otimes a^*\widehat{L}) = H^i(X, \tau_{<2}F_{e,*}\Omega^\cdot_X \otimes P \otimes a^*\widehat{L})$$

for $i \leq 1$. Moreover, since X lifts to $W_2(k)$ and $\mathrm{char}(k) \geq 2 > 1$, by Theorem 2.21, we have in $D(X)$ an isomorphism

$$\tau_{<2}F_*\Omega^\cdot_X \cong \bigoplus_{i<2} \Omega^i_X[-i].$$

Then

$$0 = H^i(X, F_{e,*}\Omega^\cdot_X \otimes P \otimes a^*\widehat{L}) = \bigoplus_l H^{i-l}(X, F_{e-1,*}\Omega^l_X \otimes P \otimes a^*\widehat{L})$$

for $i \leq 1$. By descending induction on e, we know that $H^{i-l}(X, \Omega^l_X \otimes P \otimes a^*\widehat{L}) = 0$ for $i \leq 1$. Hence, by Serre duality, we finally get $H^j(X, \Omega^i_X \otimes P \otimes a^*\widehat{L}^\vee) = 0$ for any $i + j \geq 3$ and $i \geq 0$. The second statement follows from Theorem 2.3 and Theorem 3.2. $\square$

CHAPTER 4

CHARACTERIZATION OF ABELIAN VARIETIES FOR LOG PAIRS

In this chapter, we work over the complex number field $\mathbb{C}$.

Definition 4.1. Let (X, Δ) be a pair and $a : X \dashrightarrow Y$ a rational map from X to another variety Y. We say that *the non-lc locus of* (X, Δ) *dominates* $a(X)$ if there is a divisor E of discrepancy < -1 that dominates $a(X)$.

The following theorem is the main theorem of this chapter.

Theorem 4.2. *Let X be a normal complex projective variety, (X, Δ) a pair such that $\kappa(K_X + \Delta) = 0$. Suppose that the Albanese map $\mathfrak{a}_X : X \dashrightarrow \mathfrak{Alb}(X)$ (the Albanese morphism $a_X : X \to \mathrm{Alb}(X)$) of X is not an algebraic fiber space. Then the non-lc locus of (X, Δ) dominates $\mathfrak{a}_X(X)$ ($a_X(X)$).*

An immediate consequence of Theorem 4.2 is

Corollary 4.3. *Let (X, Δ) be a projective lc pair. Assume that $\kappa(K_X + \Delta) = 0$ and the dimension of $\mathfrak{Alb}(X)$ or $\mathrm{Alb}(X)$ is equal to the dimension of X. Then X is birational to an abelian variety.*

Lemma 4.4. *Suppose that $\mathrm{char}(k) = 0$. Let (X, Δ) be a log canonical pair and $f : X \to A$ an algebraic fiber space from X to an abelian variety A. If $\kappa(K_{F_f} + \Delta|_{F_f}) \geq 0$, then $\kappa(K_X + \Delta) \geq 0$.*

Proof. If $\kappa(K_{F_f} + \Delta|_{F_f}) \geq 0$, then $H^0(F_f, \mathcal{O}_{F_f}(m(K_{F_f} + \Delta|_{F_f}))) \neq 0$ for a sufficiently divisible m such that $m(K_X + \Delta)$ is Cartier. So by cohomology and base change, we have that $f_*(\mathcal{O}_X(m(K_X + \Delta))) \neq 0$.

Next we prove that $V^0(f_*\mathcal{O}_X(m(K_X + \Delta))) \neq 0$. By Theorem 3.4, we know that $f_*\mathcal{O}_X(m(K_X + \Delta))$ is a GV sheaf, in particular

$$V^i(f_*\mathcal{O}_X(m(K_X + \Delta))) \supseteq V^{i+1}(f_*\mathcal{O}_X(m(K_X + \Delta)))$$

for all $i \geq 0$. Suppose that $V^0(f_*\mathcal{O}_X(m(K_X + \Delta))) = 0$. Then $V^i(f_*\mathcal{O}_X(m(K_X + \Delta))) = 0$

for all i. By cohomology and base change, we have $R\hat{S}(f_*\mathcal{O}_X(m(K_X+\Delta)))=0$ (cf. [36, Proof of Proposition 2.13]). So by [27, Theorem 2.2], we have $f_*(\mathcal{O}_X(m(K_X+\Delta)))=0$, which contradicts what we deduced in the last paragraph.

So we have that $V^0(f_*\mathcal{O}_X(m(K_X+\Delta)))\neq 0$. On the other hand, by [35, Theorem 1.3], $V^0(f_*\mathcal{O}_X(m(K_X+\Delta)))$ is a finite union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(A)$. In particular, we have

$$0\neq H^0(A,f_*\mathcal{O}_X(m(K_X+\Delta))\otimes P)=H^0(X,\mathcal{O}_X(m(K_X+\Delta))\otimes f^*P)$$

for some torsion elements $P\in\mathrm{Pic}^0(A)$. This implies that $H^0(X,\mathcal{O}_X(m'(K_X+\Delta)))\neq 0$ for sufficiently divisible m', hence $\kappa(K_X+\Delta)\geq 0$. □

Lemma 4.5. *Let $f:X\to Z$ be a morphism of varieties. Let E be a divisor over X and $g:Z'\to Z$ a birational morphism. Then E dominates Z if and only if E dominates Z'.*

Proof. Obvious. □

Proof of Theorem 4.2. We first deal with the case of the Albanese map. The case of Albanese morphism is almost the same and will be explained at the end of the proof. Let $\mu:Y\to X$ be a log resolution of (X,Δ) and we can write

$$K_Y+\tilde{\Delta}=\mu^*(K_X+\Delta)+E_+-E_-$$

where $\tilde{\Delta}$ is the strict transform of Δ and E_+ and E_- are effective exceptional divisors of μ with no common components. Denote $D:=\tilde{\Delta}+E_-$ and for convenience, we denote the Albanese morphism of Y by a as well. We have $\kappa(K_Y+D)=\kappa(K_X+\Delta)=0$ (cf. [29, Lemma II.3.11]). Let

$$Y\xrightarrow{g}Z\xrightarrow{h}a_Y(Y)\subseteq\mathrm{Alb}(Y)\tag{4.1}$$

be the Stein factorization of a_Y.

Lemma 4.6. *If the Albanese map of Z is an algebraic fiber space, then so is that of Y.*

Proof. After possibly passing to a smooth model of Z, we can assume that the Albanese map of Z is a morphism. By universality, h factors through the Albanese morphism a_Z of Z, and $a_Z\circ g$ factors through a_Y. So we have the following diagram:

$$\begin{tikzcd}
Z \arrow[r, "a_Z"] \arrow[dr, "h"] & \mathrm{Alb}(Z) \arrow[d, shift right, "\mathrm{t}"'] \\
Y \arrow[u, "g"] \arrow[r, "a_Y"] & \mathrm{Alb}(Y) \arrow[u, shift right, "\mathrm{s}"']
\end{tikzcd}$$

By construction, we have that

- $s \circ t \circ a_Z \circ g = s \circ h \circ g = s \circ a_Y = a_Z \circ g$;

- $t \circ s \circ a_Y = t \circ a_Z \circ g = h \circ g = a_Y$.

Since g is surjective, we see that $s \circ t = \mathrm{id}_{a_Z(Z)}$ and $t \circ s = \mathrm{id}_{a_Y(Y)}$. Since $a_Z(Z)$ and $a_Y(Y)$ generate $\mathrm{Alb}(Z)$ and $\mathrm{Alb}(Y)$, respectively, we know that s and t are isomorphisms. Moreover, since g and a_Z are both algebraic fiber spaces, we are done. $\square$

Suppose that no components of D with coefficients ≥ 1 dominate $a_Y(Y)$, or equivalently, Z. By Lemma 4.5 and [29, Lemma II.3.11], we can assume Y is smooth. By [17, Theorem 13], there is an étale cover $q : Z' \to Z$ such that $Z' \cong B \times W$ where B is an abelian variety, W is birational to a smooth variety which is of general type and of maximal Albanese dimension. Let $Y' := Y \times_Z Z'$. We then do a resolution $\nu : V \to W$ and let Z'' and Y'' be the corresponding base changes. Let $\rho : Y^\sharp \to Y$ be a log resolution of $(Y'', r_*^{-1}D)$, where $r_*^{-1}D$ is the strict transform of D' on Y''. The situation is as follows.

$$\begin{tikzcd}
Y^\sharp \arrow[r, "\rho"] & Y'' \arrow[r, "g''"] \arrow[d, "r"] & Z'' = B \times V \arrow[r, "p_V"] \arrow[d, "s"] & V \arrow[d, "\nu"] \\
& Y' \arrow[r, "g'"] \arrow[d, "p"] & Z' = B \times W \arrow[r, "p_W"] \arrow[d, "q"] & W \\
& Y \arrow[r, "g"] \arrow[d, "\mu"] & Z & \\
& X & &
\end{tikzcd}$$

By construction, V is smooth, of general type and of maximal Albanese dimension. We define $D' := p^*D$ and define $D^\sharp$ via the following inequality:

$$K_{Y^\sharp} + D^\sharp = (r \circ \rho)^*(K_{Y'} + D') + E,$$

where D' and E are effective and have no common components. Then $D^\sharp$ has snc support as well. Since p is an étale cover and $r \circ \rho$ is a birational morphism, by [29, Lemma II.3.11], we have

$$0 = \kappa(K_Y + D) = \kappa(K_{Y^\sharp} + D^\sharp).$$

By Lemma 4.5, no components of $D^\sharp$ with coefficient ≥ 1 dominate Z''. We denote the horizontal part of $D^\sharp$ with respect to $g'' \circ \rho$ as $(D^\sharp)^{\mathrm{hor}}$. Then the coefficients of $(D^\sharp)^{\mathrm{hor}}$ are in $[0,1]$. By [2, Theorem 4.2 and Remark 4.3], we know that

$$\kappa(K_{Y^\sharp} + (D^\sharp)^{\mathrm{hor}}) \geq \kappa((K_{Y^\sharp} + (D^\sharp)^{\mathrm{hor}})|_{F_{Y^\sharp/V}}) + \kappa(V). \tag{4.2}$$

and $\kappa(V) \geq 0$ as V is of general type.

On the other hand, we have

$$\kappa(K_{F_{Y^\sharp/Z''}} + (D^\sharp)^{\mathrm{hor}}|_{F_{Y^\sharp/Z''}}) = \kappa(K_{F_{Y^\sharp/Z''}} + D^\sharp|_{F_{Y^\sharp/Z''}}) \geq 0.$$

So by Lemma 4.4, $\kappa((K_{Y^\sharp} + (D^\sharp)^{\mathrm{hor}})|_{F_{Y^\sharp/V}}) \geq 0$, and by (4.2), we have $\kappa(K_{Y^\sharp} + (D^\sharp)^{\mathrm{hor}}) \geq 0$. We also have $\kappa(K_{Y^\sharp} + D^\sharp) = 0$, which forces that $\kappa(K_{Y^\sharp} + (D^\sharp)^{\mathrm{hor}}) = 0$. Again by (4.2), we obtain $\kappa(V) = 0$. But V is of general type by construction, so V, hence W, has to be a point. Now we consider the composition of maps

$$B \xrightarrow{q} Z \xrightarrow{h} \mathrm{Alb}(Y).$$

Since q and h are finite and $h(Z)$ generates $\mathrm{Alb}(Y)$, the composite $h \circ q$ has to be an isogeny, hence Z is birational to an abelian variety. Finally, by Lemma 4.6, we are done.

For the case of the Albanese moprhism, we just do the Stein factorization as in (4.1) for a_X without taking the resolution, and the rest of the proof is the same. $\square$

CHAPTER 5

ON THE CLASSIFICATION OF SURFACES OF GENERAL TYPE IN POSITIVE CHARACTERISTIC WITH EULER CHARACTERISTIC 1 AND ALBANESE DIMENSION 4

We begin this section by giving an upper bound for genus and irregularity of surfaces with Euler characteristic 1. We denote by $p_g(X)$ and $q(X)$ the geometric genus and the irregularity of X, respectively, (see Definition 2.22) and write p_g and q for short if no confusion can be made.

Proposition 5.1. *Let X be a minimal projective surface of general type over k such that $p_g \geq 2$ and* $\mathrm{char}(k) > 0$. *Then*

$$K_X^2 \geq 2p_g + q - 4. \tag{5.1}$$

If, moreover, X is liftable to $W_2(k)$ and $\chi(\mathcal{O}_X) = 1$, then $p_g = q \leq 4$.

Proof. First we would like to prove (5.1). (5.1) is well known to experts, but we include a proof for the benifit of the reader. Since we assume $p_g \geq 2$, we can write $|K_X| = M + Z$, where M is a linear series that has no fixed divisors and Z is the fixed part. Note that members in M are not necessarily smooth. We can take $D \in M$ and consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D|_D) \to 0 \tag{5.2}$$

which yields a long exact sequence

$$0 \to H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X(D)) \xrightarrow{r_1} H^0(\mathcal{O}_D(D|_D)) \to \ldots \tag{5.3}$$

After twisting (5.2) by $\mathcal{O}_X(D)$ and taking the long exact sequence, we have

$$0 \to H^0(\mathcal{O}_X(D)) \to H^0(\mathcal{O}_X(2D)) \xrightarrow{r_2} H^0(\mathcal{O}_D(2D|_D)) \to \ldots \tag{5.4}$$

Now we prove that $\dim(\mathrm{Im}(r_2)) \geq 2\dim(\mathrm{Im}(r_1)) - 1$. We have a homomorphism $r : \mathrm{Im}(r_1) \oplus \mathrm{Im}(r_1) \to \mathrm{Im}(r_2)$ induced by the commutativity of the following diagram

$$\begin{tikzcd} H^0(\mathcal{O}_X(D)) \otimes H^0(\mathcal{O}_X(D)) \arrow[r, "r_1 \otimes r_1"] \arrow[d] & H^0(\mathcal{O}_D(D|_D)) \otimes H^0(\mathcal{O}_D(D|_D)) \arrow[d] \\ H^0(\mathcal{O}_X(2D)) \arrow[r, "r_2"] & H^0(\mathcal{O}_D(2D|_D)) \end{tikzcd}$$

When we view those images as linear systems, the map r_1 is finite, because divisors on a curve are points, and there are only finitely many ways to separate these points into two parts. Therefore, we have

$$\dim(\mathrm{Im}(r_2)) - 1 \geq 2(\dim(\mathrm{Im}(r_1)) - 1), \tag{5.5}$$

which is exactly what we want. Next, by Riemann-Roch,

$$h^0(\mathcal{O}_X(2K_X)) = \chi(\mathcal{O}_X) + K_X^2 = p_g - q + 1 + K_X^2. \tag{5.6}$$

By (5.3), (5.4) and (5.5), we know that

$$\begin{aligned} h^0(\mathcal{O}_X(2K_X)) - h^0(\mathcal{O}_X(K_X)) &= \dim(\mathrm{Im}(r_2)) \\ \geq 2\dim(\mathrm{Im}(r_1)) - 1 &= 2h^0(\mathcal{O}_X(K_X)) - 3. \end{aligned}$$

Then by (5.6),

$$p_g - q + 1 + K_X^2 = h^0(\mathcal{O}_X(2K_X)) \geq 3h^0(\mathcal{O}_X(K_X)) - 3 = 3p_g - 3.$$

Therefore, we have proved (5.1). If $\chi(\mathcal{O}_X) = 1$ and X is liftable to $W_2(k)$, then by Proposition 2.25, we have $9 \geq K_X^2 \geq 2p_g + q - 4$ and Proposition 5.1 follows easily. $\square$

In this section, we will consider surfaces that satisfy the following condition:

($\star$) X is a projective surfaces that is mAd and lifts to $W_2(k)$, its Picard variety has no supersingular factors and $\chi(\mathcal{O}_X) = 1$.

The main theorem of this section is as follows.

Theorem 5.2. *Let X be a smooth minimal projective surface of general type over an algebraically closed field k of characteristic ≥ 11 that satisfies ($\star$). Denote the Albanese morphism as $a : X \to A$ and asssume that a is separable. If $\dim(A) = 4$, then $X = C_1 \times C_2$ where C_1 and C_2 are smooth curves and $g(C_1) = g(C_2) = 2$.*

Remark 5.3. By the main result of [15], we know $\dim(A) \leq q(X)$, so by Proposition 5.1, $\dim(A) \leq 4$ and $\dim(A) = 4$ implies $p_g(X) = q(X) = 4$.

Lemma 5.4. *Let X be a projective surface which is mAd and lifts to $W_2(k)$, and its Picard variety has no supersingular factors. Then either X admits an irrational pencil of genus $\geq \dim(V^1(\omega_X)) \geq 1$ or $\dim V^1(\omega_X) = 0$.*

Proof. Suppose that X does not admit an irrational pencil of genus $\geq \dim(V^1(\omega_X))$ and $\dim V^1(\omega_X) \geq 1$. Let $a : X \to A$ be the Albanese morphism. By [32, Corollary 3.4], $V^1(\omega_X)$ is completely linear, so we can take $T + Q$ to be a component of maximal dimension of $V^1(\omega_X)$ where T is an abelian subvariety of dimension ≥ 1 and Q is a torsion element. Now T is an abelian subvariety of $\hat{A}$, so after taking its dual, we get a surjective morphism $c : A \to \hat{T}$. Denote $c \circ a$ by g. If $\dim(g(X)) = 1$, then as $g(X)$ generates $\hat{T}$, its genus must be $\geq \dim(T) = \dim(V^1(\omega_X)) \geq 1$. Since we have supposed this is not the case, we must have $\dim(g(X)) = 2$. By Theorem 3.6 and Theorem 2.3, we know that

$$h^1(X, \omega_X \otimes Q \otimes g^*R) = h^1(\hat{T}, g_*(\omega_X \otimes Q) \otimes R) = 0$$

for general $R \in \mathrm{Pic}^0(\hat{T}) = T$, which contradicts the definition of $T + Q$. □

Next we will show that under the condition of the above theorem, X has at least two distinct fibrations onto curves of certain genera.

Proposition 5.5. *Let X be a smooth minimal projective surface over k of positive characteristic that satisfies ($\star$). If $\dim(A) = 4$, then $\dim(V^1(\omega_X)) \geq 1$. In particular, X admits an irrational pencil.*

Proof. Suppose that this is not the case. By Lemma 5.4, $\dim(V^1(\omega_X)) = 0$, so $h^0(X, \omega_X \otimes P) = 1$ for all but finitely many $P \in \mathrm{Pic}^0(X)$. We also know that $a_*\omega_X$ is GV_0, so we have

$$\begin{aligned} R^0\widehat{S}(D_A(a_*\omega_X)) &= R\widehat{S}(D_A(a_*\omega_X)) \\ &= R\widehat{S}(D_A \circ Ra_*(\omega_X)) = R\widehat{S}(Ra_* \circ D_X(\omega_X)) \\ &= R\widehat{S}(Ra_*R\mathcal{H}om(\omega_X, \omega_X[2])) = R\hat{S}Ra_*\mathcal{O}_X[2], \end{aligned}$$

where the first equality is by Theorem 3.2, the second equality is by Theorem 2.3 and the third equality is by Grothendieck duality. This means that $R\hat{S}Ra_*\mathcal{O}_X$ is a sheaf in degree

2. We claim that $R\hat{S}Ra_*\mathcal{O}_X[2] = L \otimes \mathcal{I}_Z$ where L is a line bundle and Z is a 0-dimensional subvariety of X.

Next we prove the claim. We first prove that $R\hat{S}Ra_*\mathcal{O}_X[2]$ is torsion-free. Denote $R\hat{S}Ra_*\mathcal{O}_X[2]$ by $\mathcal{F}$. At the beginning of the proof, we have deduced that $h^0(X, \omega_X \otimes P) = 1$ for all but finitely many $P \in \mathrm{Pic}^0(X)$, and since $\chi(\mathcal{O}_X) = 1$, we see that $h^i(X, \omega_X \otimes P) = 0$ also for all but finitely many $P \in \mathrm{Pic}^0(X)$ and any $i > 0$. So by cohomology and base change, $\mathcal{F}$ is a line bundle except for a finite number of points. By Theorem 2.9, the following equality holds

$$(-1_A)^* Ra_*\mathcal{O}_X[-4] = RSR\hat{S}(Ra_*\mathcal{O}_X) = RS(\mathcal{F}[-2]), \tag{5.7}$$

which means $R^0S(\mathcal{F}) = R^1S(\mathcal{F}) = 0$ and $R^2S(\mathcal{F}) = a_*\mathcal{O}_X$. On the other hand, we have the following exact sequence

$$0 \to T \to \mathcal{F} \to \mathcal{F}^{\vee\vee} \to Q \to 0 \tag{5.8}$$

where T is supported on finitely many points. So when we consider the long exact sequence

$$0 \to R^0S(T) \to R^0S(\mathcal{F}) \to ...$$

we get $R^0S(T) = 0$ because $R^0S(\mathcal{F}) = 0$ as above, and since T is supported on finite many points, we have $T = 0$ and then $\mathcal{F}$ is torsion-free. Since $\mathcal{F}^{\vee\vee}$ is a reflexive sheaf of rank 1 on a smooth variety, then $\mathcal{F}^{\vee\vee} = L$ is a line bundle, hence $\mathcal{F} = L \otimes \mathcal{I}_Z$ where Z is a 0-dimensional subscheme.

Now we consider the following short exact sequence:

$$0 \to L \otimes \mathcal{I}_Z \to L \to L \otimes \mathcal{O}_Z \to 0,$$

which yields a long exact sequence

$$0 \to R^0S(L \otimes \mathcal{I}_Z) \to R^0S(L) \to R^0S(L \otimes \mathcal{O}_Z) \to R^1S(L \otimes \mathcal{I}_Z) \to ... \tag{5.9}$$

Among these terms $R^0S(L \otimes \mathcal{I}_Z) = R^1S(L \otimes \mathcal{I}_Z) = 0$ and $R^2S(L \otimes \mathcal{I}_Z) = a_*\mathcal{O}_X$ as deduced above (immediately after (5.7)), $R^iS(L \otimes \mathcal{O}_Z) = 0$ for $i \geq 1$ because $L \otimes \mathcal{O}_Z$ is supported on a finite number of points, and $R^iS(L) \neq 0$ only for $i = i(L)$, the index of L. Denote $R^0S(L \otimes \mathcal{O}_Z)$ by V which is a vector bundle in degree 0. We get contradiction by considering the following two cases:

Case 1. $Z = \emptyset$, which is equivalent to $V = 0$. In this case, $i(L) = 2$ so $R^2S(L) = R^2S(L \otimes \mathcal{I}_Z) = a_*\mathcal{O}_X$. This means that the support of $R^2S(L)$ is $a(X)$. But on the other hand, by Lemma 2.17, the support of $R^2S(L)$ must be an abelian subvariety, and since $\dim(a(X)) = 2$, we know that $a(X)$ does not generate A. Contradiction.

Case 2. $Z \neq \emptyset$, then V is a nonzero vector bundle and $R^2S(L \otimes \mathcal{I}_Z) = a_*\mathcal{O}_X$ which is also nonzero. Thus $R^iS(L) \neq 0$ for $i = 0$ and $i = 2$ which is impossible as $R^j(L) = 0$ for any $j \neq i(L)$. □

Proposition 5.6. *Assume that X satisfies all the conditions in Proposition 5.5, then there are two irrational pencils on X over two smooth curves C_1, C_2 satisfying either one of the following conditions*

1. *both $g(C_1)$ and $g(C_2)$ are ≥ 2,*

2. *one of $g(C_1)$ and $g(C_2)$ is ≥ 3 and the other is 1,*

such that the induced morphism $X \to C_1 \times C_2$ is generically finite.

Proof. First by Remark 3.7 and Proposition 5.5, we have $1 \leq \dim(V^1(\omega_X)) \leq 3$. We take an irreducible component of maximal dimension in $V^1(\omega_X)$ and denote it as $Q_0 + E$, where Q_0 is a torsion element and E is an abelian subvariety. By Poincaré's complete reducibility theorem, we have an isogeny $E \times F \to \hat{A}$ where F is an abelian subvariety of dimension $\dim(A) - \dim(V^1(\omega_X)) = 4 - \dim(V^1(\omega_X))$. After dualizing this map, we get the dual isogeny $b : A \to \hat{F} \times \hat{E}$. For convenience, we denote $A_1 = \hat{E}$, $A_2 = \hat{F}$ and denote $pr_{\hat{E}} \circ b \circ a$ and $pr_{\hat{F}} \circ b \circ a$ by a_1 and a_2, respectively.

Next we prove that X admits two dominant morphisms onto two smooth curves whose geometric genera satisfy (1) or (2) in Proposition 5.6, and the induced morphism from X to their product is generically finite. We consider the following three cases:

Case 1. If $\dim(V^1(\omega_X)) = 1$, then $\dim(A_1) = 1$ and $\dim(A_2) = 3$. We then prove that a_2 induces an irrational pencil of genus ≥ 3. Let

$$\tilde{V^1} = \{(P, Q) \in \hat{A}_1 \times \hat{A}_2 | h^1(X, \omega_X \otimes a_1^*P \otimes a_2^*Q) \neq 0\},$$

then there is a finite map $\tilde{V^1} \to V^1 = V^1(\omega_X)$ which implies that $\dim(\tilde{V^1}) = 1$. Let P be a general element in $\mathrm{Pic}^0(A_1)$, and let

$$S_P^1 = \{Q \in \mathrm{Pic}^0(A_2) | (P, Q) \in \tilde{V}^1\}.$$

Then $S_P^1 = \tilde{V}^1 \cap (P \times \hat{A}_2)$ is the fiber of the projection map $\tilde{V}^1 \to \hat{A}_1$ over P. So we have

$$1 = \dim(\tilde{V}^1) = \dim(S_P^1) + \dim(\hat{A}_1) = \dim(S_P^1) + 1$$

which forces $\dim(S_P^1)$ to be 0.

If a_2 is not generically finite, then a_2 factors through $a(X)$ and the map $a(X) \to a_2(X)$ is an elliptic fibration onto its image. Since $a_2(X)$ generates A_2, X is fibered over a curve of genus at least 3.

So next we would like to assume that a_2 is generically finite and then derive a contradiction. Since $\dim(S_P^1) = 0$, we have that

$$h^0(X, \omega_X \otimes a_1^*P \otimes a_2^*Q) = h^0(A_2, a_{2,*}(\omega_X \otimes a_1^*P) \otimes Q) = 1$$

for all but finitely many $Q \in \mathrm{Pic}^0(A_2)$. By Theorem 3.6, $a_{2,*}(\omega_X \otimes a_1^*P)$ is GV_0, so we have

$$\begin{aligned}
&R^0\widehat{S}(D_{A_2}(a_{2,*}(\omega_X \otimes a_1^*P))) = R\widehat{S}(D_{A_2}(a_{2,*}(\omega_X \otimes a_1^*P))) \\
=&R\widehat{S}(D_{A_2} \circ Ra_{2,*}(\omega_X \otimes a_1^*P)) = R\widehat{S}(Ra_{2,*} \circ D_X(\omega_X \otimes a_1^*P)) \\
=&R\widehat{S}(Ra_{2,*}R\mathcal{H}om(\omega_X \otimes a_1^*P, \omega_X[2])) = R\hat{S}Ra_{2,*}(a_1^*P^\vee)[2],
\end{aligned}$$

where the first equality is by Theorem 3.2, the second equality is by Theorem 2.3 and the third equality is by Grothendieck duality. This means $R\hat{S}Ra_{2,*}(a_1^*P^\vee)$ is a sheaf in degree 2. If we denote $R\hat{S}Ra_{2,*}(a_1^*P^\vee)[2]$ by $\mathcal{G}$, then by Theorem 2.9, we know

$$(-1_{\hat{A}_2})^*Ra_{2,*}(a_1^*P^\vee)[-3] = RS(R\hat{S}Ra_{2,*}(a_1^*P^\vee)) = RS(\mathcal{G}[-2]),$$

which means $R^0S(\mathcal{G}) = 0$ and $R^1S(\mathcal{G}) = a_{2,*}(a_1^*P^\vee)$. Then arguing as in the proof of Proposition 5.5 starting from (5.8) with $\mathcal{F}$ replaced by $\mathcal{G}$, we see that $\mathcal{G} = L \otimes \mathcal{I}_Z$ where L is a line bundle and Z is supported on a finite set.

Now we can construct a long exact sequence as (5.9). In the long exact sequence, we have $R^0S(L \otimes \mathcal{I}_Z) = 0$, $R^1S(L \otimes \mathcal{I}_Z) = a_{2,*}(a_1^*P^\vee)$, $R^iS(L) \neq 0$ only for $i = i(L)$ and $R^iS(L \otimes \mathcal{O}_Z) = 0$ for $i \geq 1$.

Next we deduce contradiction for all $i(L)$. Since L is supported on $\hat{A}_2$, by Proposition 2.17, we see that $i(L) \leq \dim(\hat{A}_2) = 3$. So we consider $i(L) = 3, 2, 1, 0$, respectively.

If $i(L) = 2$ or 3, we can get $V = R^1S(L \otimes \mathcal{I}_Z) = a_{2,*}(a_1^*P^\vee)$ where $V := R^0S(L \otimes \mathcal{O}_Z)$. But since V is a vector bundle and $a_{2,*}(a_1^*P^\vee)$ is a torsion sheaf, this is a contradiction.

If $i(L) = 1$, we have a short exact sequence

$$0 \to V \to a_{2,*}(a_1^*P^\vee) \to \widehat{L} \to 0$$

which forces $V = 0$ because $a_{2,*}(a_1^*P^\vee)$ is a torsion sheaf. Then the support of $\widehat{L}$ is $a_2(X)$. We already know from Proposition 2.17 that the support of $\widehat{L}$ is an abelian subvariety, and on the other hand, according to the above construction of a_2, we know $a_2(X)$ is 2-dimensional and generates A_2. This is a contradiction.

If $i(L) = 0$, then we have a short exact sequence

$$0 \to \widehat{L} \to V \to a_{2,*}(a_1^*P^\vee) \to 0. \tag{5.10}$$

After taking its dual, we get a long exact sequence

$$\begin{aligned} 0 &\to a_{2,*}(a_1^*P^\vee)^\vee \to V^\vee \to \widehat{L}^\vee \\ &\to \mathcal{E}xt^1(a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}) \to \mathcal{E}xt^1(V, \mathcal{O}_{A_2}) \to \dots \end{aligned} \tag{5.11}$$

In (5.11), we have $\mathcal{E}xt^1(V, \mathcal{O}_{A_2}) = 0$ as V is locally free and $a_{2,*}(a_1^*P^\vee)^\vee = 0$ as $a_{2,*}(a_1^*P^\vee)$ is a torsion sheaf. Then (5.11) reduces to a short exact sequence

$$0 \to V^\vee \to \widehat{L}^\vee \to \mathcal{E}xt^1(a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}) \to 0. \tag{5.12}$$

Since $\widehat{L} \neq 0$, by (5.10), we see that $V \neq 0$. Since $L \otimes \mathcal{O}_Z$ is supported on points, by [27, Example 2.9], we know $V^\vee = \bigoplus_i V_i$ where V_i is a successive extension by elements in $\mathrm{Pic}^0(A_2)$. If we consider one step of such extension as the exact sequence

$$0 \to W' \to W \to R \to 0$$

where $R \in \mathrm{Pic}^0(A_2)$, then after twisting it by $R^\vee$, it becomes

$$0 \to W' \otimes R^\vee \to W \otimes R^\vee \to \mathcal{O}_{A_2} \to 0.$$

Then $H^3(A_2, \mathcal{O}_{A_2}) \neq 0$ implies that $H^3(A_2, W \otimes R^\vee) \neq 0$. Following such successive extension, we finally get that there exists $P' \in \mathrm{Pic}^0(A_2)$ such that

$$H^3(A_2, V^\vee \otimes P') \neq 0. \tag{5.13}$$

Moreover, by Theorem 2.9, we have

$$R\hat{S}RS(L) = (-1_{\hat{A}_2})^*L[-3],$$

then

$$R^i\hat{S}(\widehat{L}) = 0, \forall i \neq 3.$$

So by cohomology and base change,

$$H^i(A_2, \widehat{L} \otimes Q) = 0, \forall i \neq 3, Q \in \mathrm{Pic}^0(A_2),$$

in particular

$$H^3(A_2, \widehat{L}^\vee \otimes P') = H^0(A_2, \widehat{L} \otimes P'^\vee) = 0.$$

This together with (5.13) and (5.12) implies that

$$H^2(A_2, \mathcal{E}xt^1(a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}) \otimes P') \neq 0. \tag{5.14}$$

On the other hand, by Grothendieck duality, we have

$$R\mathcal{H}om(Ra_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}[3]) = Ra_{2,*}(R\mathcal{H}om(a_1^*P^\vee, \omega_X[2])),$$

hence

$$R\mathcal{H}om(Ra_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2})[1] = Ra_{2,*}(R\mathcal{H}om(a_1^*P^\vee, \omega_X)), \tag{5.15}$$

By Theorem 2.3, the right side of (5.15) is just $a_{2,*}(\omega_X \otimes a_1^*P)$, so after taking cohomology of (5.15) in degree 0, we have

$$\mathcal{E}xt^1(Ra_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}) = a_{2,*}(\omega_X \otimes a_1^*P). \tag{5.16}$$

Now by [14, (3.7)], we have the following spectral sequence

$$E_2^{p,q} := \mathcal{E}xt^p(R^{-q}a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}) \Rightarrow \mathcal{E}xt^{p+q}(Ra_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}). \tag{5.17}$$

Since a_2 is generically finite onto its image, we see that $R^{-q}a_{2,*}(a_1^*P^\vee)$ is 0 for $q \leq -2$ and $q \geq 1$, and $R^1a_{2,*}(a_1^*P^\vee)$ is supported on points. Hence $\mathcal{E}xt^p(R^{-q}a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2})$ is 0 when $q = -1$ and $p \neq 3$, and $\mathcal{E}xt^3(R^1a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2})$ is supported on points. Therefore, the spectral sequence (5.17) degenerates at $E_3^{p,q}$ and $\mathcal{E}xt^1(Ra_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2})$ is equal to the kernel of the following natural morphism in the second page

$$d_2^{1,0} : E_2^{1,0} = \mathcal{E}xt^1(a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}) \to E_2^{3,-1} = \mathcal{E}xt^3(R^1a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}).$$

So by (5.16), we have the following short exact sequence

$$0 \to a_{2,*}(\omega_X \otimes a_1^*P) \to \mathcal{E}xt^1(a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}) \to \mathcal{E}xt^3(R^1a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}) \to 0. \quad (5.18)$$

Twisting (5.18) by P' constructed in (5.13), we have

$$\begin{aligned} 0 \to a_{2,*}(\omega_X \otimes a_1^*P) \otimes P' \to \mathcal{E}xt^1(a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}) \otimes P' \\ \to \mathcal{E}xt^3(R^1a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}) \otimes P' \to 0. \end{aligned} \quad (5.19)$$

By (5.14) and the fact that $\mathcal{E}xt^3(R^1a_{2,*}(a_1^*P^\vee), \mathcal{O}_{A_2}) \otimes P'$ is supported on points, we know that $H^2(A_2, a_{2,*}(\omega_X \otimes a_1^*P) \otimes P') \neq 0$. On the other hand,

$$H^2(A_2, a_{2,*}(\omega_X \otimes a_1^*P) \otimes P') = H^2(X, \omega_X \otimes a_1^*P \otimes a_2^*(P')) = H^0(X, a_1^*P^\vee \otimes a_2^*(P'^\vee)),$$

but since $P^\vee$ is a general element of A_1, we have $a_1^*P^\vee \otimes a_2^*P'^\vee \neq \mathcal{O}_X$, so $H^0(X, a_1^*P^\vee \otimes a_2^*(P'^\vee)) = 0$ and we get a contradiction.

So we have deduced that a_2 induces an irrational pencil of genus ≥ 3. On the other hand, since A_1 is an elliptic curve, a_1 must be surjective, otherwise $a(X)$ would be contained in a 3-dimensional abelian subvariety of A which is impossible. Therefore, in this case, a_1 and a_2 induce two dominant morphims we want.

Case 2. If $\dim(V^1(\omega_X)) = 3$, then $\dim(A_1) = 3$ and $\dim(A_2) = 1$. By the proof of Lemma 5.4, a_1 induces an irrational pencil of genus ≥ 3 and by the argument at the end of *Case 1*, a_2 induces a dominant morphism onto a curve of geometric genus ≥ 1. So in this case, we also have the two dominant morphisms we claimed.

Case 3. If $\dim(V^1(\omega_X)) = 2$, then $\dim(A_1) = \dim(A_2) = 2$. We define $\tilde{V}^1$ in the same way as in *Case 1* and define

$$T_Q^1 = \{P \in \mathrm{Pic}^0(A_1) | (P, Q) \in \tilde{V}^1\}$$

for a general $Q \in \hat{A}_2$.

Next we prove that $T_Q^1 \neq \emptyset$. If $T_Q^1 = \emptyset$, then

$$h^i(X, \omega_X \otimes a_2^*Q \otimes a_1^*P) = 0, \forall P \in \mathrm{Pic}^0(A_1), i > 0, \tag{5.20}$$

so

$$h^0(A_1, a_{1,*}(\omega_X \otimes a_2^*Q) \otimes P) = h^0(X, \omega_X \otimes a_2^*Q \otimes a_1^*P) = 1, \forall P \in \mathrm{Pic}^0(A_1).$$

Consider the Leray spectral sequence

$$E_2^{ij} = H^i(A_1, R^j a_{1,*}(\omega_X \otimes a_2^*Q) \otimes P) \Rightarrow H^{i+j}(X, \omega_X \otimes a_2^*Q \otimes a_1^*P).$$

We actually have $E_2^{ij} = 0$ for $i \geq 2$ or $j \geq 2$. Indeed if a_1 is generically finite, this is by (5.20) and Theorem 2.3 and if a_1 is not generically finite, this is for dimensional reasons. So this spectral sequence degenerates at E_2, in particular

$$\begin{aligned}
&h^1(X, \omega_X \otimes a_2^*Q \otimes a_1^*P) \\
=&h^1(A_1, a_{1,*}(\omega_X \otimes a_2^*Q) \otimes P) + h^0(A_1, R^1 a_{1,*}(\omega_X \otimes a_2^*Q) \otimes P) \\
\geq&h^1(A_1, a_{1,*}(\omega_X \otimes a_2^*Q) \otimes P).
\end{aligned}$$

Hence $h^1(A_1, a_{1,*}(\omega_X \otimes a_2^*Q) \otimes P) = 0$. So by cohomology and base change, $R\hat{S}_{A_1}(a_{1,*}(\omega_X \otimes a_2^*Q))$ is a line bundle in degree 0 which we denote by L_1. By Theorem 2.9, $a_{1,*}(\omega_X \otimes a_2^*Q) = (-1_{A_1})^*\widehat{L_1}$. However, the support of $a_{1,*}(\omega_X \otimes a_2^*Q)$ is a curve which spans A_1 while by Proposition 2.17, the support of $(-1_{A_1})^*\widehat{L_1}$ is an abelian subvariety in A_1. This is a contradiction.

We also have

$$2 = \dim(\tilde{V}^1) = \dim(T_Q^1) + \dim(\hat{A}_2) = \dim(T_Q^1) + 2$$

which forces $\dim(T_Q^1)$ to be 0. This, together with $T_Q^1 \neq \emptyset$, implies that in $\tilde{V}^1 \subset \hat{A}_1 \times \hat{A}_2$, there is a 2-dimensional component which is a torsion translate of an abelian subvariety and dominates $\hat{A}_2$. We denote this component by $Q_0 + \hat{B}_1$, then $\hat{B}_1 \cap (\hat{A}_1 \times \{0\})$ is a finite number of points. This means that the natural homomorphism $\hat{A}_1 \times \hat{B}_1 \to \hat{A}_1 \times \hat{A}_2$ is an isogeny (cf. proof of [28, p.160 Theorem 1]). We denote the following composition of isogenies

$$\hat{A}_1 \times \hat{B}_1 \to \hat{A}_1 \times \hat{A}_2 \to \hat{A}$$

by $\hat{\varphi}$. Then we have the dual isogeny $\varphi : A \to A_1 \times B_1$. By the proof of Lemma 5.4, each of the two morphisms $X \to A_1$ and $X \to B_1$ gives a dominant morphism to a curve of geometric genus ≥ 2. So in this case, what we claimed also holds.

Therefore, by the argument for the above three cases, we have constructed an isogeny $A \to A_1 \times A_2$ where $(\dim(A_1), \dim(A_2)) = (3,1)$ or $(2,2)$ (by symmetry we can assume that $\dim(A_1) \geq \dim(A_2)$). Each of the projections $\psi_i : A \to A_i$ induces a morphism $g_i : X \to a(X) \to \tilde{C}_i$ where $\tilde{C}_i$ is a smooth curve (we can pass to their normalization if necessary because their geometric genera do not change), and the genera of the two curves $(g(\tilde{C}_1), g(\tilde{C}_2))$ can be either $(m,1)$ or (n,k) where $m \geq 3$ and $n,k \geq 2$. However, so far, g_i may not satisfy $g_{i,*}\mathcal{O}_X = \mathcal{O}_{\tilde{C}_i}$. Denote the kernel of ψ_i by K_i and the connected component containing the origin by K_i^0. By [26, Proposition 5.31], $(K_i^0)_{\mathrm{red}}$ is an abelian subvariety of A, so the quotient $A/(K_i^0)_{\mathrm{red}}$ exists and the quotient morphism $A \to A/(K_i^0)_{\mathrm{red}}$ is separable. For convenience, we also use A_i to denote $A/(K_i^0)_{\mathrm{red}}$ and use φ_i and $h_i : X \to C_i$ to denote the quotient morphism $A \to A/(K_i^0)_{\mathrm{red}}$ and the Stein factorization of $\varphi_i \circ a$, respectively, and by passing to normalization, we can assume that each C_i is smooth. By construction, we have that $h_{i,*}\mathcal{O}_X = \mathcal{O}_{C_i}$ and g_i factors through h_i, so by the Hurwitz formula, the genera of $(g(C_1), g(C_2))$ are also either $(m,1)$ or (n,k) where $m \geq 3$ and $n,k \geq 2$. So h_1 and h_2 are the two irrational pencils we want. □

Proof of Theorem 5.2. Let $h_i : X \to C_i$ be the two irrational pencils constructed above. WLOG we assume that $g(C_1) \geq g(C_2)$, in particular $g(C_1) \geq 2$. We have the following lemma:

Lemma 5.7. *There is an injective morphism of sheaves*

$$\omega_{C_i} \otimes h_{i,*}\omega_X \to h_{i,*}\omega_X^2,$$

where $i = 1, 2$.

Proof. By the above construction, we have the following diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "h_i"] & a(X) \arrow[r] & A \arrow[d, "\varphi_i"] \\
& C_i \arrow[r, "j_i"] & A_i
\end{tikzcd}$$

Now by the separability of φ_i we get the following injective homomorphisms induced by pullbacks

$$H^0(A_i, \Omega^1_{A_i}) \to H^0(A, \Omega^1_A). \tag{5.21}$$

Since $H^0(A_i, \Omega^1_{A_i}) \times \mathcal{O}_{A_i} \cong \Omega^1_{A_i} = \mathcal{T}^\vee_{A_i}$, we have $H^0(A_i, \Omega^1_{A_i}) \cong T^\vee_{A_i,e}$. Then we have

$$H^0(A, \Omega^1_A) \cong H^0(A_1, \Omega^1_{A_1}) \oplus H^0(A_2, \Omega^1_{A_2}) \tag{5.22}$$

and the isomorphism is induced by pullback via $\varphi_1 \times \varphi_2$. Then, after wedging $H^0(A, \Omega^1_A)$ with itself, by (5.22), we get

$$\begin{aligned} &H^0(A, \Omega^2_A) = \wedge^2 H^0(A, \Omega^1_A) \\ &= \wedge^2 H^0(A_1, \Omega^1_{A_1}) \oplus \wedge^2 H^0(A_2, \Omega^1_{A_2}) \oplus (H^0(A_2, \Omega^1_{A_2}) \otimes H^0(A_1, \Omega^1_{A_1})). \end{aligned}$$

By separability of a (note that this is the only place where we use separability of a), there exists $\omega \in H^0(A, \Omega^2_A)$ such that $0 \neq a^*\omega \in H^0(X, \Omega^2_X)$ on X. If $\omega \in \wedge^2 H^0(A_i, \Omega^1_{A_i})$, then since the above diagram commutes if we go through the pullback by $j_i \circ h_i$, then ω must go to 0 as C_i is 1-dimensional. Therefore, there are two 1-forms $\omega_i \in H^0(A_i, \Omega^1_{A_i})$ such that $a^*(\varphi_1 \times \varphi_2)^*(\omega_1 \boxtimes \omega_2) \neq 0$. This also implies that

$$H^0(C_i, \omega_{C_i}) \otimes H^0(A_j, \Omega^1_{A_j}) \to H^0(X, \omega_X)$$

is nonzero, hence

$$h_i^*\omega_{C_i} \otimes H^0(A_j, \Omega^1_{A_j}) \to \omega_X \tag{5.23}$$

induced by pullbacks is nonzero for $(i,j) = (1,2)$ or $(2,1)$. This means that there exists a $\omega'_j \in H^0(A_j, \Omega^1_{A_j})$ such that the morphism

$$h_i^*\omega_{C_i} \xrightarrow{\wedge\omega'_j} \omega_X \tag{5.24}$$

is nonzero, hence by Lemma 2.27, it is injective.

Finally, after twisting (5.24) by ω_X and pushing it forward by h_i and using the projection formula, we are done. $\square$

Next we would like to prove that under the condition of Theorem 5.2, a general fiber F of h_1 is smooth and to do this, we need to estimate $p_a(F)$. By the adjunction formula, we have

$$\omega_F = (\omega_X + F)|_F = \omega_X|_F,$$

where ω_F is the dualizing sheaf of F. So by [11, Theorem III.12.8 and Corollary III.12.9] and Serre duality,

$$\mathrm{rk}(h_{1,*}\omega_X) = h^0(F, \omega_X|_F) = h^0(F, \omega_F) = h^1(F, \mathcal{O}_F).$$

Then by Riemann-Roch theorem on curves, we have

$$\begin{aligned} &\chi(\omega_{C_1} \otimes h_{1,*}\omega_X) = \chi(h_{1,*}\omega_X) + \mathrm{rk}(h_{1,*}\omega_X)(2g(C_1) - 2) \\ =&\chi(h_{1,*}\omega_X) + p_a(F)(2g(C_1) - 2) \geq \chi(h_{1,*}\omega_X) + 2p_a(F). \end{aligned} \tag{5.25}$$

Next we make two observations:

1. We estimate the left-hand side as follows:

$$\begin{aligned} &\chi(\omega_{C_1} \otimes h_{1,*}\omega_X) \leq h^0(C_1, \omega_{C_1} \otimes h_{1,*}\omega_X) \leq h^0(C_1, h_{1,*}(\omega_X^2)) \\ =&h^0(X, \omega_X^2) = \chi(\mathcal{O}_X) + K_X^2 \leq 1 + 9 = 10 \end{aligned}$$

where the second and the third inequalities are by Lemma 5.7 and Proposition 2.25, respectively.

2. We claim that $\chi(h_{1,*}\omega_X) \geq 0$. To prove this, consider Leray spectral sequence $E_2^{p,q} = H^p(C_1, R^q h_{1,*}\omega_X) \Rightarrow H^{p+q}(X, \omega_X)$. Since $E_2^{p,q} = 0$ for $p \geq 2$ for dimensional reasons, we know that the spectral sequence degenerates at E_2. That means

$$4 = h^1(X, \omega_X) = h^0(C_1, R^1 h_{1,*}\omega_X) + h^1(C_1, h_{1,*}\omega_X) \geq h^1(C_1, h_{1,*}\omega_X).$$

Finally, we have $\chi(h_{1,*}\omega_X) = h^0(h_{1,*}\omega_X) - h^1(h_{1,*}\omega_X) \geq 4 - 4 = 0$.

Therefore, in (5.25), we know $10 \geq 2p_a(F)$, so $p_a(F) \leq 5$. Then Tate's theorem (cf. [24, Theorem 5.1]) implies that if F is singular, then $(p-1)/2$ divides $p_a(F) - p_a(\tilde{F})$, in particular $(p-1)/2 \leq 5$. So when $p \geq 13$, F is smooth. When $p = 11$ and F is singular, then it can only happen that $p_a(F) = 5$ and $p_a(\tilde{F}) = 0$. But then F is rational which is contradictory to the assumption that X is of mAd. So when $p = 11$, F is also smooth.

Since X is mAd and of general type, by Proposition 2.26, $g(F)$ must be no less than 2. So by a theorem of Arakelov (see [1, Théorème d'Arakelov and Corollaire]), we know

$$9 \geq K_X^2 \geq 8(g(C_1)-1)(g(F)-1) \geq 8(2-1)(2-1) = 8.$$

This forces that $g(C_1) = g(F) = 2$, so we see that the case $g(C_1) \geq 3$ cannot happen, in particular $g(C_1) = g(C_2) = 2$. Then after we divide the morphism $F \to C_2$ into a separable one and a purely inseparable one, by Hurwitz's formula, we have $F \cong C_2$ (cf. [11, Ch.IV Example 2.5.4 and Proposition 2.5]), and when we restrict h_2 to any fiber of h_1, say X_t for $t \in C_1$, it is a composition of Frobenius morphisms F^{e_t}. Since for any $p \in C_2$, we get that $h_2^* \mathcal{O}_{C_2}(p) \cdot X_t$ is constant with respect to $t \in C_1$. So $e_t = e_1$ is constant with respect to t. We also do all the above argument for $h_2 : X \to C_2$ and get that the general fiber of h_2 is isomorphic to C_1 and h_1 induces F^{e_2} on these fibers for a uniform e_2.

Denote the induced morphism $X \to C_1 \times C_2$ by f. There is a generically étale morphism $\phi : C_1' \to C_1$ such that $X \times_{C_1} C_1' = C_1' \times C_2$ and the induced map $X \times_{C_1} C_1' \to C_1'$ is the projection onto C_1' (cf. [16] right after Definition 2.3), where C_1' is a projective curve. We denote the induced map $X \times_{C_1} C_1' = C_1' \times C_2 \to X$ by φ and the projections onto C_1' and C_2 by p_1 and p_2, respectively.

Now by the above construction, φ and ϕ are both separable morphisms and $\deg(\varphi) = \deg(\phi)$. By construction of h_2, we know that $h_2 \circ \varphi$ is also separable and contracts every $C_1' \times \{c\}$ where c is a general closed point of C_2. So by the Hurwitz formula $h_2 \circ \varphi$ restricted to every $\{c'\} \times C_2$, where c' is a general closed point of C_1', is a composition of Frobenius morphisms and an automorphism of C_2, in particular on the underlying topological space, it is an automorphism of C_2. Since $\mathrm{Aut}(C_2)$ is finite, there is a uniform automorphism $\alpha : C_2 \to C_2$ such that on the underlying topological spaces, $h_2 \circ \varphi = \alpha \circ p_2$. Now we do a Stein factorization of $h_2 \circ \varphi$ which we denote by

$$C_1' \times C_2 \xrightarrow{r} C_2 \xrightarrow{\beta} C_2.$$

The situation is as follows.

$$\begin{tikzcd}
 & C_2 \arrow[r, "\beta"] & C_2 \\
C_1' \times C_2 \arrow[ur, "r"] \arrow[r, "\varphi"] \arrow[d, "p_1"] & X \arrow[r, "f"] \arrow[ur, "h_2"] \arrow[dr, "h_1"'] & C_1 \times C_2 \arrow[u] \arrow[d] \\
C_1' \arrow[rr, "\phi"] & & C_1
\end{tikzcd}$$

Since $h_2 \circ \varphi$ is separable, we know that β is separable and hence $\beta \in \mathrm{Aut}(C_2)$. By definition of Stein factorization, we have $(\beta \circ r)_* \mathcal{O}_{C_1' \times C_2} = \mathcal{O}_{C_2}$, and we have seen above that the map of the underlying topological spaces for $\beta \circ r$ is $\alpha \circ p_2$. So $h_2 \circ \varphi = \alpha \circ p_2$ as a morphism of schemes. On the other hand $\varphi \circ h_1 = \phi \circ p_1$, so $f \circ \varphi = (\phi \circ p_1) \times (\alpha \circ p_2)$ and in particular, it is separable with degree $\deg(\phi)$, which is equal to the degree of φ. This implies that the degree of f is 1, hence f is birational. Finally, by smoothness and minimality of X, we have $X = C_1 \times C_2$. $\square$

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